

# Classical integrable systems and soliton equations related to eleven-vertex R-matrix

A. Levin<sup>♭♯</sup>   M. Olshanetsky<sup>♯♯</sup>   A. Zotov<sup>◇♯♯</sup>

<sup>♭</sup> – NRU HSE, Department of Mathematics, Myasnitskaya str. 20, Moscow, 101000, Russia

<sup>♯</sup> – ITEP, B. Cheremushkinskaya str. 25, Moscow, 117218, Russia

<sup>♯</sup> – MIPT, Institutskii per. 9, Dolgoprudny, Moscow region, 141700, Russia

◇ – Steklov Mathematical Institute RAS, Gubkina str. 8, Moscow, 119991, Russia

E-mails: alevin@hse.ru, olshanet@itep.ru, zotov@mi.ras.ru

## Abstract

In our recent paper we suggested a natural construction of the classical relativistic integrable tops in terms of the quantum  $R$ -matrices. Here we study the simplest case – the 11-vertex  $R$ -matrix and related  $gl_2$  rational models. The corresponding top is equivalent to the 2-body Ruijsenaars-Schneider (RS) or the 2-body Calogero-Moser (CM) model depending on its description. We give different descriptions of the integrable tops and use them as building blocks for construction of more complicated integrable systems such as Gaudin models and classical spin chains (periodic and with boundaries). The known relation between the top and CM (or RS) models allows to re-write the Gaudin models (or the spin chains) in the canonical variables. Then they assume the form of  $n$ -particle integrable systems with  $2n$  constants. We also describe the generalization of the top to 1+1 field theories. It allows us to get the Landau-Lifshitz type equation. The latter can be treated as non-trivial deformation of the classical continuous Heisenberg model. In a similar way the deformation of the principal chiral model is also described.

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## 1 Introduction

In this paper we deal with the quantum 11-vertex  $R$ -matrix [5]<sup>1</sup>:

$$R^h(z) = \begin{pmatrix} \hbar^{-1} + z^{-1} & 0 & 0 & 0 \\ -\hbar - z & \hbar^{-1} & z^{-1} & 0 \\ -\hbar - z & z^{-1} & \hbar^{-1} & 0 \\ -\hbar^3 - 2z\hbar^2 - 2\hbar z^2 - z^3 & \hbar + z & \hbar + z & \hbar^{-1} + z^{-1} \end{pmatrix} \quad (1.1)$$

**Relativistic integrable tops.** It was recently observed [20] that (1.1) is the simplest example of the quantum rational  $\mathfrak{gl}_N$   $R$ -matrix appearing from the classical relativistic integrable top. The relativistic top is defined by its classical Lax operator in terms of the quantum  $R$ -matrix<sup>2</sup>

$$L^\eta(z, \mathcal{S}) \equiv L^\eta(z) = \text{tr}_2 (R_{12}^\eta(z) \mathcal{S}_2) , \quad \mathcal{S} = \text{Res}_{z=0} L^\eta(z) , \quad \mathcal{S}_2 = 1 \otimes \mathcal{S} \quad (1.2)$$

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<sup>1</sup>See also [26], where it was derived by non-trivial limiting procedure from the Baxter quantum elliptic  $R$ -matrix [2].

<sup>2</sup>Notice that the Planck constant in the  $R$ -matrix is replaced by the relativistic deformation parameter  $\eta$  of the Ruijsenaars-Schneider (RS) type [22]. In the RS model it equals to the ratio of the coupling constant to the light speed.

with the spectral parameter  $z$ . The dynamical variables are the components of  $2 \times 2$  matrix  $\mathcal{S} \in \mathfrak{gl}_2$ , which is the residue of  $L^\eta(z, \mathcal{S})$ . The Poisson structure is generated by the quadratic  $r$ -matrix structure

$$\{L_1^\eta(z), L_2^\eta(w)\} = [L_1^\eta(z) L_2^\eta(w), r_{12}(z-w)], \quad (1.3)$$

where  $r_{12}(z-w)$  is the classical  $r$ -matrix (the classical limit of (1.1))<sup>3</sup>:

$$r_{12}(z) = \begin{pmatrix} 1/z & 0 & 0 & 0 \\ -z & 0 & 1/z & 0 \\ -z & 1/z & 0 & 0 \\ -z^3 & z & z & 1/z \end{pmatrix} \quad (1.4)$$

The equations of motion are of the Euler type:

$$\dot{\mathcal{S}} = [\mathcal{S}, J^\eta(\mathcal{S})], \quad (1.5)$$

where the inverse inertia tensor  $J^\eta$  is given by (2.15), (2.52).

In the non-relativistic limit  $\eta \rightarrow 0$  the  $r$ -matrix structure (1.3) becomes the linear:

$$\{L_1(z), L_2(w)\} = [L_1(z) + L_2(w), r_{12}(z-w)]. \quad (1.6)$$

Similarly to (1.2) the Lax matrix is expressed in terms of the classical  $r$ -matrix

$$L(z, S) \equiv L(z) = \text{tr}_2(r_{12}(z)S_2), \quad S = \text{Res}_{z=0} L(z) \in \mathfrak{gl}_2. \quad (1.7)$$

It leads to the following equations:

$$\dot{S} = [S, J(S)]. \quad (1.8)$$

While the Lax matrix (1.2) is the quasi-classical limit of the quantum  $L$ -operator, the standard description of the classical quadratic Poisson structures deals with the different Lax matrix:

$$\tilde{L}(z, \tilde{S}) = \tilde{S}_0 \mathbf{1}_{2 \times 2} + L(z, \tilde{S}) - \frac{1}{2} \text{tr} L(z, \tilde{S}) \mathbf{1}_{2 \times 2}, \quad \tilde{S}_0 = \text{tr} \tilde{S} / 2, \quad \tilde{S} \in \mathfrak{gl}_2. \quad (1.9)$$

The latter is independent of  $\eta$ . Similarly to the non-relativistic case it is defined in terms of the classical  $r$ -matrix. It provides the rational analogue of the classical Sklyanin algebra in its original form [23]. The relation between two descriptions ( $\eta$ -dependent (1.2) and  $\eta$ -dependent (1.9)) comes from both – the limit  $\eta \rightarrow 0$  and from the explicit change of variables:

$$\begin{aligned} \mathcal{S}(\eta, \tilde{S}) &= \frac{1}{2} \tilde{L}\left(\frac{\eta}{2}, \tilde{S}\right), \\ L^\eta\left(z - \frac{\eta}{2}, \tilde{L}\left(\frac{\eta}{2}, \tilde{S}\right)\right) &= \frac{\text{tr} L^\eta\left(z - \frac{\eta}{2}, \tilde{S}\right)}{\text{tr} \tilde{S}} \tilde{L}(z, \tilde{S}). \end{aligned} \quad (1.10)$$

See details in next Section. Both descriptions based on the quadratic Poisson brackets ((1.2) and (1.9)) can be considered as top-like forms of the rational 2-body Ruijsenaars-Schneider (RS) model [22], while the linear is the top-like form of the 2-body Calogero-Moser (CM) model [4]. We will show that the equation (1.8) describes both - CM and RS models. The difference is in the Poisson structures and the Hamiltonians.

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<sup>3</sup>The non-relativistic rational  $\mathfrak{gl}_N$  tops were described in [1], while the  $\mathfrak{sl}_2$  case was derived previously in [3].

**Limit to XXX case.** In order to get the standard XXX  $R$ -matrices consider the following deformations of (1.1):

$$R^{\hbar, \epsilon}(z) = \epsilon R^{\epsilon \hbar}(\epsilon z), \quad r^{\epsilon}(z) = \epsilon r(\epsilon z), \quad (1.11)$$

i.e.

$$r_{12}^{\epsilon}(z) = \begin{pmatrix} 1/z & 0 & 0 & 0 \\ -z\epsilon^2 & 0 & 1/z & 0 \\ -z\epsilon^2 & 1/z & 0 & 0 \\ -z^3\epsilon^4 & z\epsilon^2 & z\epsilon^2 & 1/z \end{pmatrix} \quad (1.12)$$

and similarly for the quantum  $R$ -matrix. Then

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} R^{\hbar, \epsilon}(z) &= R^{\text{xxx}}(z) = \frac{1}{\hbar} 1 \otimes 1 + \frac{1}{z} P_{12}, \\ \lim_{\epsilon \rightarrow 0} r^{\epsilon}(z) &= r^{\text{xxx}}(z) = \frac{1}{z} P_{12}. \end{aligned} \quad (1.13)$$

The Lax matrices (1.2), (1.7) and (1.9) are written in terms of the quantum and classical  $R$ -matrices. These Lax operators are the building blocks for more complicated integrable systems (see below). Then the limit  $\epsilon \rightarrow 0$  describes transition to the XXX-type models for all the systems considered in this paper. The constant parameter  $\epsilon$  can be treated as a coupling constant in integrable tops because the standard XXX case corresponds to free motion  $\dot{S} = 0$ .

**Spin chains and Gaudin models.** Having the quadratic Poisson structure (1.3) the classical periodic spin chain with  $n$  sites is naturally defined [8] via the transfer matrix

$$T(z, \mathcal{S}^1, \dots, \mathcal{S}^n) = T(z) = L^{\eta_1}(z - z_1, \mathcal{S}^1) \dots L^{\eta_n}(z - z_n, \mathcal{S}^n), \quad (1.14)$$

where  $z_k$  are the inhomogeneities parameters. In our case (1.2) it has the quasi-classical form:

$$T_0(z) = \text{tr}_{1 \dots n} (R_{01}^{\eta_1}(z - z_1) \dots R_{0n}^{\eta_n}(z - z_n) (\mathcal{S}^1)_1 \dots (\mathcal{S}^n)_n), \quad (1.15)$$

where the index 0 is for the "auxiliary" space of the classical Lax representation. In the non-relativistic limit  $\eta \rightarrow 0$  it gives rise to the Lax operator of the Gaudin model:

$$L_0^{\text{G}}(z) = \sum_{a=1}^n \text{tr}_a (r_{0a}(z - z_a) S^a) \stackrel{(1.7)}{=} \sum_{a=1}^n L_0(z - z_a, S^a). \quad (1.16)$$

To construct the finite chain one needs to have solutions of the reflection equations [24]. While the  $\eta$ -independent Lax matrix (1.9) satisfies the standard reflection equation (3.19), the  $\eta$ -dependent (1.2) requires a small modification due to (1.10):

$$\begin{aligned} \{L_1^{\eta}(z), L_2^{\eta}(w)\} &= \frac{1}{2} [L_1^{\eta}(z) L_2^{\eta}(w), r_{12}(z - w)] - \\ &- \frac{1}{2} L_1^{\eta}(z) r_{12}(z + w + \eta) L_2^{\eta}(w) + \frac{1}{2} L_2^{\eta}(w) r_{12}(z + w + \eta) L_1^{\eta}(z). \end{aligned} \quad (1.17)$$

We consider the Gaudin models and spin chains in Section 3.

**1+1 models and soliton equations.** For the homogeneous  $z_k = 0$  spin chain (1.15) the continuous limit leads to the 1+1 field theories, which are integrable in the sense of the classical

inverse scattering method [31]. The equations of motion are generated by (the zero curvature condition) the Zakharov-Shabat equations [31]:

$$\partial_t U - k \partial_x V = [U, V], \quad (1.18)$$

where  $U$  and  $V$  are  $\mathfrak{gl}_2$ -valued functions on the circle (with the coordinate  $x$ ). They also depend on the spectral parameter and dynamical fields  $S(x)$ . It was shown in [16] that the mechanical (0+1) models described by non-dynamical  $r$ -matrix can be generalized to 1+1 field theory (1.18) straightforwardly: one should simply use the same Lax operator (1.7):

$$U^{\text{LL}}(z, S(x)) = L(z, S(x)) = \text{tr}_2(r_{12}(z)S_2(x)). \quad (1.19)$$

As will be shown in Section 4.1 it leads to Landau-Lifshitz [15, 25] type equation:

$$\partial_t S = \alpha[S, S_{xx}] + [S, J(S)], \quad (1.20)$$

where  $S_{xx} = \partial_x^2 S$ ,  $J(S)$  is the same as in the top case and  $\alpha$  is a constant. In the light of the Symplectic Hecke Correspondence [16] this type of the Landau-Lifshitz model is equivalent to the (rational  $\mathfrak{sl}_2$ ) 1+1 Calogero field theory [11, 16].

In the same way we consider the 2-poles case

$$U^{\text{chiral}}(z, S(x)) = L(z - z_1, S^1(x)) + L(z - z_2, S^2(x)). \quad (1.21)$$

and get the principal chiral model [30, 6, 8, 10] in the form:

$$\begin{cases} \partial_t S^1 - k \partial_x S^1 = -2[S^1, L(z_1 - z_2, S^2)], \\ \partial_t S^2 + k \partial_x S^2 = -2[S^2, L(z_1 - z_2, S^1)]. \end{cases} \quad (1.22)$$

At last, we describe the 1+1 Gaudin models which can be regarded as the interacting Landau-Lifshitz magnets. See Section 4.

**Purpose of the paper.** In this paper we study relationships between the simplest examples of 2-body systems of Calogero-Moser model and Ruijsenaars-Schneider and their relations to integrable tops. We give accurate description for all cases and show that the models can be described in a similar way. As by product we notice that top's description allows naturally to deal with the reflection equation which makes possible to construct finite chains on a lattice. At last we mention that our description is adequate for constructing 1+1 field generalizations such as principal chiral models and interacting Landau-Lifshitz models. The examples are non-trivial and new. When parameter  $\epsilon$  in (1.11) goes to 0 we come back to the ordinary Gaudin models.

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## 2 Relativistic Rational top

### 2.1 Three descriptions

Here we outline the algebraic structures given in [20]. They are universal, and are valid for  $\mathfrak{gl}_N$  and not only for the rational case.

We give three description of the same classical model:

1. In terms of the linear  $r$ -matrix structure (2.3) with the Lax matrix (2.4). It is gauge equivalent to the (spin) Calogero-Moser model. It is further used for constructing the Gaudin models and 1+1 field theories.
2. In terms of the quadratic  $r$ -matrix structure (2.5) and  $\eta$ -independent Lax matrix (2.6). It is the conventional form of the classical Sklyanin algebra. It is used for constructing the classical spin chains.
3. In terms of the quadratic  $r$ -matrix structure (2.2) and  $\eta$ -dependent Lax matrix (2.1). It is gauge equivalent to the quantum  $R$ -matrix and the (spin) Ruijsenaars-Schneider model. The first two descriptions can be obtained from it by the limit  $\eta \rightarrow 0$ , and the second is also related to it explicitly (see (2.7)-(2.8)). It can be also used for constructing the spin chains.

As it was mentioned in the Introduction the relativistic top is defined by the classical Lax operator

$$L^\eta(z, \mathcal{S}) \equiv L^\eta(z) = \text{tr}_2 (R_{12}^\eta(z) \mathcal{S}_2) , \quad \mathcal{S} = \text{Res}_{z=0} L(z) , \quad (2.1)$$

where  $R_{12}^\eta(z)$  is the quantum  $R$ -matrix (1.1), and the quadratic  $r$ -matrix structure

$$\{L_1^\eta(z), L_2^\eta(w)\} = [L_1^\eta(z) L_2^\eta(w), r_{12}(z-w)] , \quad (2.2)$$

where  $r_{12}(z-w)$  is the classical  $r$ -matrix (1.4).

The non-relativistic limit  $\eta \rightarrow 0$  gives rise to the linear  $r$ -matrix structure

$$\{L_1(z), L_2(w)\} = [L_1(z) + L_2(w), r_{12}(z-w)] . \quad (2.3)$$

and the Lax operator

$$L(z, S) \equiv L(z) = \text{tr}_2 (r_{12}(z) S_2) , \quad S = \text{Res}_{z=0} L(z) . \quad (2.4)$$

The non-relativistic model is bihamiltonian. In the elliptic case it was observed in [14]. The linear  $r$ -matrix structure (2.3) is compatible with the quadratic one (2.2)

$$\{\tilde{L}_1(z), \tilde{L}_2(w)\} = [\tilde{L}_1(z) \tilde{L}_2(w), r_{12}(z-w)] , \quad (2.5)$$

but with the different Lax operator

$$\begin{aligned} \tilde{L}(z, \tilde{S}) &= \tilde{S}_0 \, 1_{2 \times 2} + L(z, \tilde{S}) - \frac{1}{2} \text{tr} L(z, \tilde{S}) \, 1_{2 \times 2} , \\ \tilde{S}_0 &= \text{tr} \tilde{S} , \quad \text{Res}_{z=0} \tilde{L}(z, \tilde{S}) = \tilde{S} - \frac{1}{2} \text{tr} \tilde{S} \, 1_{2 \times 2} . \end{aligned} \quad (2.6)$$

The latter is independent of  $\eta$  and provides the rational analogue of the classical Sklyanin algebra in its original form [23]. It appears that the Lax matrices (2.1) and (2.6) satisfying quadratic  $r$ -matrix structures are explicitly related<sup>4</sup>:

$$L^\eta\left(z - \frac{\eta}{2}, \tilde{L}\left(\frac{\eta}{2}, \tilde{S}\right)\right) = \frac{\text{tr} L^\eta\left(z - \frac{\eta}{2}, \tilde{S}\right)}{\text{tr} \tilde{S}} \tilde{L}(z, \tilde{S}) \quad (2.7)$$

There is an explicit change of variables relating  $\eta$ -dependent and  $\eta$ -independent descriptions:

$$\mathcal{S}(\eta, \tilde{S}) = \frac{1}{2} \tilde{L}\left(\frac{\eta}{2}, \tilde{S}\right). \quad (2.8)$$

The coefficient  $1/2$  in the r.h.s. is not fixed by (2.7). We choose this normalization factor in order to have  $\text{Res}_{\eta=0} \mathcal{S}(\eta, \tilde{S}) = \text{Res}_{z=0} \tilde{L}(z, \tilde{S})$ .

Let us remark here that the first description is naturally related to the (spin) Calogero-Moser (CM) model while the third is related to the (spin) Ruijsenaars-Schneider (RS) model [22]. In the spinless case relation is given by the explicit change of variables generated by the special gauge transformations acting on the Lax operators (see [1] and [20]). In view of (2.8) the second description is also related to the RS model. In the same time, the first and the second descriptions lead to the same equations of motion (see (2.13) below). Hence, *the top's equations of motion can be treated as the common description for the RS and CM models*. We discuss this point in Section 2.5.

It was shown in [20] that there is a number of interrelations between Lax pairs in different descriptions. For example, the expansion of the  $\eta$ -dependent Lax operator (2.1) near  $\eta = 0$  provides  $M$ -operators for all three descriptions:

$$L^\eta(z, \mathcal{S}) = \eta^{-1} \frac{\text{tr} \mathcal{S}}{2} 1_{2 \times 2} - M(z, \mathcal{S}) + \eta \mathcal{M}(z, \mathcal{S}) + O(\eta^2). \quad (2.9)$$

The coefficient  $M(z, \mathcal{S}) = -L(z, \mathcal{S})$  is the  $M$ -operator for  $L^\eta(z, \mathcal{S})$ :

$$\dot{L}^\eta(z, \mathcal{S}) = [L^\eta(z, \mathcal{S}), M(z, \mathcal{S})]. \quad (2.10)$$

The next term ( $\mathcal{M}(z, \mathcal{S})$ ) in (2.9) is the  $M$ -operator for the Lax matrices (2.4) and (2.6) (the  $M$ -operators are the same since the Lax matrices are differed by only scalar terms)

$$\dot{L}(z, \mathcal{S}) = [L(z, \mathcal{S}), \mathcal{M}(z, \mathcal{S})]. \quad (2.11)$$

$$\dot{\tilde{L}}(z, \tilde{S}) = [\tilde{L}(z, \tilde{S}), \tilde{M}(z, \tilde{S})] = [\tilde{L}(z, \tilde{S}), \mathcal{M}(z, \tilde{S})].$$

The Lax equations (2.10) and (2.11) give rise to equations of motion of Euler type:

$$\dot{\mathcal{S}} = [\mathcal{S}, J^\eta(\mathcal{S})] \quad (2.12)$$

and

$$\begin{aligned} \dot{S} &= [S, J(S)], \\ \dot{\tilde{S}} &= [\tilde{S}, J(\tilde{S})]. \end{aligned} \quad (2.13)$$

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<sup>4</sup>The shift by  $\eta/2$  is specific for the rational case. In the elliptic case it is  $\eta$ . This difference comes from the normalization  $z \rightarrow z/N$  for the rational spectral parameter.

respectively. The inverse inertia tensor  $J(S)$  in (2.13) can be found from (2.11):

$$J : S \rightarrow J(S) = \mathcal{M}(0, S), \quad (2.14)$$

while  $J^\eta(\mathcal{S})$  is of the form<sup>5</sup>:

$$J^\eta : \mathcal{S} \rightarrow J^\eta(\mathcal{S}) = \text{tr}_2 \left( \left( R_{12}^{\eta, (0)} - r_{12}^{(0)} \right) \mathcal{S}_2 \right), \quad (2.15)$$

where  $R_{12}^{\eta, (0)}$  and  $r_{12}^{(0)}$  are the coefficients of the local expansion of (1.1) and (1.4) near  $z = 0$ <sup>6</sup>:

$$\begin{aligned} R_{12}^h(z) &= \sum_{k=-1}^{\infty} z^k R_{12}^{h, (k)} = \frac{1}{z} P_{12} + R_{12}^{h, (0)} + z R_{12}^{h, (1)} + O(z^2), \\ r_{12}(z) &= \frac{1}{z} P_{12} + r_{12}^{(0)} + O(z). \end{aligned} \quad (2.16)$$

Notice that plugging the change of variables (2.8) into equations of motion (2.12) we get the Lax equations for the  $\eta$ -independent description

$$\partial_t \tilde{L}(\frac{\eta}{2}, \tilde{S}) = \frac{1}{2} [\tilde{L}(\frac{\eta}{2}, \tilde{S}), J^\eta(\tilde{L}(\frac{\eta}{2}, \tilde{S}))], \quad (2.17)$$

where  $\eta/2$  plays the role of the spectral parameter (i.e. (2.17) is identity in  $\eta$ ). In this way we get an alternative definition of  $M$ -operator for the  $\eta$ -independent description:

$$\tilde{M}(z, \tilde{S}) = \frac{1}{2} J^{2z}(\tilde{L}(z, \tilde{S})). \quad (2.18)$$

Indeed, one can verify that (cf. (2.11))

$$\frac{1}{2} J^{2z}(\tilde{L}(z, \tilde{S})) = \mathcal{M}(z, \tilde{S}) + \frac{1}{2z} \tilde{S}_0 \mathbf{1}_{2 \times 2}. \quad (2.19)$$

## 2.2 Non-relativistic description

**Equations of motion.** Consider the Lie coalgebra  $\mathfrak{gl}_2^*$  with coordinates  $S_{ij}$

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad (2.20)$$

and Poisson-Lie brackets

$$\{S_{ij}, S_{kl}\} = \delta_{il} S_{kj} - \delta_{kj} S_{il}, \quad i, j, k, l = 1, 2. \quad (2.21)$$

The Casimir functions are defined by

$$C_1 = \text{tr} S = S_{11} + S_{22}, \quad C_2 = \frac{1}{2} \text{tr} S^2 = \frac{1}{2} (S_{11}^2 + S_{22}^2 + S_{12} S_{21}). \quad (2.22)$$

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<sup>5</sup>It would be interesting to find out its mechanical treatment among the known integrable examples of the rigid body motion (see reviews [28]).

<sup>6</sup>In the  $\mathfrak{gl}_2$  case, which is under consideration, (2.15) is simplified since  $r_{12}^{(0)} = 0$  for (1.4).



By fixation of  $C_{1,2}$  the space  $\mathfrak{gl}_2^*$  (2.20) reduces to a coadjoint orbit of  $\mathrm{GL}_2$  Lie group. Such orbit is the phase space of the model. The fixation of  $C_{1,2}$  does not change the brackets (2.21). On a surface  $C_{1,2} = \text{const}$  the brackets are non-degenerated.

Consider the Hamiltonian function

$$H = -S_{12}(S_{11} - S_{22}). \quad (2.23)$$

It generates equations of motion

$$\begin{cases} \dot{S}_{11} = -\dot{S}_{22} = -S_{12}(S_{11} - S_{22}), \\ \dot{S}_{21} = -2S_{12}S_{21} + (S_{11} - S_{22})^2, \\ \dot{S}_{12} = 2S_{12}^2. \end{cases} \quad (2.24)$$

The latter can be written in the top-like form

$$\dot{S} = \{H, S\} = [S, J(S)], \quad (2.25)$$

where the inverse inertia tensor  $J$  is the following linear functional on  $\mathfrak{gl}_2$ :

$$J(S) = - \begin{pmatrix} S_{12} & 0 \\ S_{11} - S_{22} & -S_{12} \end{pmatrix} \quad (2.26)$$

With this notation the Hamiltonian (2.23) acquires the form of the Euler top:

$$H = \frac{1}{2} \text{tr} (S J(S)). \quad (2.27)$$

**Lax pair.** The Lax matrix equals

$$L(z, S) = \frac{1}{z} \begin{pmatrix} S_{11} - z^2 S_{12} & S_{12} \\ S_{21} - z^2(S_{11} - S_{22}) - z^4 S_{12} & S_{22} + z^2 S_{12} \end{pmatrix} \quad (2.28)$$

It has the form

$$L(z) = \frac{1}{z} L^{(-1)} + z L^{(1)} + z^3 L^{(3)}, \quad L^{(-1)} := S, \quad (2.29)$$

i.e. it is skew-symmetric

$$L(z) = -L(-z) \quad (2.30)$$

The generating function for the Hamiltonian(s) is given by

$$\frac{1}{2} \text{tr} L^2(z) = \frac{1}{z^2} C_2 + 2H, \quad (2.31)$$

where  $C_2$  is the Casimir function (2.22) and  $H$  is (2.23). The Lax equations

$$\dot{L}(z) = \{H, L(z)\} = [L(z), \mathcal{M}(z)] \quad (2.32)$$

with

$$\mathcal{M}(z) = - \begin{pmatrix} S_{12} & 0 \\ S_{11} - S_{22} + 2z^2 S_{12} & -S_{12} \end{pmatrix} \quad (2.33)$$

reproduce equations of motion (2.24). Note that the linear combination

$$\tilde{\mathcal{M}}(z) = \frac{1}{z}L(z) - \mathcal{M}(z) = \frac{1}{z^2}S + z^2 \begin{pmatrix} 0 & 0 \\ S_{12} & 0 \end{pmatrix}. \quad (2.34)$$

have a simple form and is equivalent to  $-\mathcal{M}(z)$  in the Lax equations.

Evaluating the residue of the Lax equation (2.32) at  $z = 0$  we get  $\dot{S} = [S, M(0)]$ . Therefore, it follows from (2.25) that the inverse inertia tensor and  $M$ -operator are related as

$$J(S) = \mathcal{M}(0), \quad (2.35)$$

where  $J(S)$  is defined by (2.26). Another simple expression for  $J(S)$  follows from the expansion (2.29). Taking into account (2.27)

$$J(S) = L^{(1)}. \quad (2.36)$$

Finally, let us give one more useful formula (that can be verified directly):

$$L(z, L(z, S)) = \frac{1}{z^2}S + 2J(S). \quad (2.37)$$

**Classical  $r$ -matrix** allows us to write all the Poisson brackets between matrix elements of the Lax matrix in the form

$$\sum_{i,j,k,l} E_{ij} \otimes E_{kl} \{L_{ij}(z), L_{kl}(w)\} := \{L_1(z), L_2(w)\} = [L_1(z) + L_2(w), r_{12}(z - w)], \quad (2.38)$$

where in  $\mathfrak{gl}_2$  case  $L_1 = L \otimes 1 = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{matrix} 1_{2 \times 2} \\ 1_{2 \times 2} \end{matrix}$ ,  $L_2 = 1 \otimes L = \begin{pmatrix} L & 0_{2 \times 2} \\ 0_{2 \times 2} & L \end{pmatrix}$ . See, for example, [8]. In our case the classical  $r$ -matrix equals

$$r_{12}(z) = \begin{pmatrix} 1/z & 0 & 0 & 0 \\ -z & 0 & 1/z & 0 \\ -z & 1/z & 0 & 0 \\ -z^3 & z & z & 1/z \end{pmatrix} \quad (2.39)$$

It can be computed from the quantum one (1.1) by the classical limit  $\lim_{\hbar \rightarrow 0} (R^\hbar(z) - \hbar^{-1} 1 \otimes 1)$ . The  $r$ -matrix satisfies the classical Yang-Baxter equation

$$[r_{12}(z - w), r_{13}(z)] + [r_{12}(z - w), r_{23}(w)] + [r_{13}(z), r_{23}(w)] = 0 \quad (2.40)$$

and have the following properties:

$$r_{12}(z) = -r_{21}(-z), \quad \text{Res}_{z=0} r_{12}(z) = P_{12}, \quad (2.41)$$

where  $P_{12} = \sum_{i,j=1}^2 E_{ij} \otimes E_{ij}$  is the permutation operator. The Lax matrix and  $r$ -matrix are simply related:

$$L(z) = \text{tr}_2(r_{12}S_2). \quad (2.42)$$

The latter relation is easy to check: evaluate the residue of both parts of (2.38) at  $w = 0$ , and then at  $z = 0$ . It gives the Poisson-Lie brackets (2.21) in the form:

$$\{S_1, S_2\} = [S_2, P_{12}]. \quad (2.43)$$

Moreover, plugging (2.42) into (2.38) with the Poisson brackets (2.43), we obtain the Yang-Baxter equation (2.40).

**Limit to free motion.** In the limit (1.12)

$$\lim_{\epsilon \rightarrow 0} (\epsilon r_{12}(z\epsilon)) = \frac{1}{z} P_{12}, \quad (2.44)$$

and we get the trivial system  $\dot{S} = 0$  with  $L(z) = \frac{1}{z} \text{tr}_2(P_{12}S_2) = \frac{1}{z} S$ ,  $H = 0$ .

### 2.3 Relativistic top: $\eta$ -dependent description

From the quantum  $R$ -matrix (1.1) written as  $R_{12}^h(z) = \sum_{i,j,k,l=1}^N R_{ij,kl}^h(z) E_{ij} \otimes E_{kl}$  we obtain the following Lax matrix:

$$L^\eta(z, \mathcal{S}) = \sum_{i,j,k,l=1}^N R_{ij,kl}^\eta(z) E_{ij} \mathcal{S}_{lk} = \quad (2.45)$$

$$\frac{1}{z} \mathcal{S}_{2 \times 2} + \frac{\text{tr}(\mathcal{S})}{\eta} 1_{2 \times 2} - (z + \eta) \begin{pmatrix} \mathcal{S}_{12} & 0 \\ (\mathcal{S}_{11} - \mathcal{S}_{22}) + (\eta^2 + z^2 + \eta z) \mathcal{S}_{12} & -\mathcal{S}_{12} \end{pmatrix} \quad (2.46)$$

The Poisson brackets are defined by the quadratic  $r$ -matrix structure

$$\{L_1^\eta(z), L_2^\eta(w)\} = [L_1^\eta(z) L_2^\eta(w), r_{12}(z - w)], \quad (2.47)$$

with the rational  $r$ -matrix (2.39). The Poisson brackets are written as

$$\mathcal{A}_{h=0,\eta}^{\text{SkI}} : \quad \{\mathcal{S}_1, \mathcal{S}_2\} = [\mathcal{S}_1 \mathcal{S}_2, r_{12}^{(0)}] + [L^{\eta,(0)}(S)_1 \mathcal{S}_2, P_{12}], \quad (2.48)$$

where we use notations of the expansion

$$L^\eta(z) = \text{tr}_2(R_{12}^\eta(z) \mathcal{S}_2) = \frac{1}{z} S + L^{\eta,(0)}(S) + z L^{\eta,(1)}(S) + O(z^2). \quad (2.49)$$

Since  $r_{12}^{(0)} = 0$ , (2.48) is simplified

$$\mathcal{A}_{h=0,\eta}^{\text{SkI}} : \quad \{\mathcal{S}_1, \mathcal{S}_2\} = [J^\eta(S)_1 \mathcal{S}_2, P_{12}], \quad (2.50)$$

This gives the Poisson brackets between the components of  $\mathcal{S}$ :

$$\begin{aligned} \{\mathcal{S}_{11}, \mathcal{S}_{12}\} &= -\eta^{-1}(\mathcal{S}_{11} + \mathcal{S}_{22})\mathcal{S}_{12} + \eta\mathcal{S}_{12}^2, & \{\mathcal{S}_{22}, \mathcal{S}_{12}\} &= \eta^{-1}(\mathcal{S}_{11} + \mathcal{S}_{22})\mathcal{S}_{12} + \eta\mathcal{S}_{12}^2 \\ \{\mathcal{S}_{11}, \mathcal{S}_{21}\} &= \eta^{-1}\mathcal{S}_{21}(\mathcal{S}_{11} + \mathcal{S}_{22}) + \eta(\mathcal{S}_{11}^2 - \mathcal{S}_{11}\mathcal{S}_{22} - \mathcal{S}_{12}\mathcal{S}_{21}) + \eta^3\mathcal{S}_{11}\mathcal{S}_{12}, \\ \{\mathcal{S}_{21}, \mathcal{S}_{22}\} &= \eta^{-1}\mathcal{S}_{21}(\mathcal{S}_{11} + \mathcal{S}_{22}) - \eta(\mathcal{S}_{22}^2 - \mathcal{S}_{11}\mathcal{S}_{22} - \mathcal{S}_{12}\mathcal{S}_{21}) + \eta^3\mathcal{S}_{12}\mathcal{S}_{22}, \\ \{\mathcal{S}_{12}, \mathcal{S}_{21}\} &= -(\mathcal{S}_{11} + \mathcal{S}_{22})(\eta^{-1}(\mathcal{S}_{11} - \mathcal{S}_{22}) + \eta\mathcal{S}_{12}), \\ \{\mathcal{S}_{11}, \mathcal{S}_{22}\} &= \eta\mathcal{S}_{12}(\mathcal{S}_{11} - \mathcal{S}_{22} + \eta^2\mathcal{S}_{12}). \end{aligned} \quad (2.51)$$

They define the Poisson structure on the phase space of the relativistic top. The equations of motion have form (2.12) with

$$J^\eta(S) = L^{\eta(0)} = - \begin{pmatrix} \eta \mathcal{S}_{12} & 0 \\ \eta^3 \mathcal{S}_{12} + \eta(\mathcal{S}_{11} - \mathcal{S}_{22}) & -\eta \mathcal{S}_{12} \end{pmatrix} + \frac{\mathcal{S}_{11} + \mathcal{S}_{22}}{\eta} 1_{2 \times 2}. \quad (2.52)$$

Written in components the equation (2.12) assumes the form:

$$\begin{aligned} \dot{\mathcal{S}}_{11} &= -\eta \mathcal{S}_{12}(\mathcal{S}_{11} - \mathcal{S}_{22} + \eta^2 \mathcal{S}_{12}) = -\dot{\mathcal{S}}_{22}, \quad \dot{\mathcal{S}}_{12} = 2\eta \mathcal{S}_{12}^2, \\ \eta^{-1} \dot{\mathcal{S}}_{21} &= (\mathcal{S}_{11} - \mathcal{S}_{22})^2 - 2\mathcal{S}_{12}\mathcal{S}_{21} + \eta^2 \mathcal{S}_{11}\mathcal{S}_{12} - \eta^2 \mathcal{S}_{22}\mathcal{S}_{12}. \end{aligned} \quad (2.53)$$

The determinant of the Lax matrix (2.46) defines the Casimir functions:

$$\det L^\eta(z) = \frac{1}{z^2} \mathcal{C}_2 + \left( \frac{1}{z\eta} + \frac{1}{\eta^2} \right) \mathcal{C}_1, \quad (2.54)$$

$$\mathcal{C}_2 = \det S = \mathcal{S}_{11}\mathcal{S}_{22} - \mathcal{S}_{12}\mathcal{S}_{21}, \quad \mathcal{C}_1 = (\mathcal{S}_{11} + \mathcal{S}_{22} + \eta^2 \mathcal{S}_{12})^2 - 4\eta^2 \mathcal{S}_{12}\mathcal{S}_{22}. \quad (2.55)$$

The  $M$ -operator for the Lax equations (2.10) reproducing equations of motion (2.12) is obtained via (2.9):

$$M(z) = -\frac{1}{z} \begin{pmatrix} \mathcal{S}_{11} - z^2 \mathcal{S}_{12} & \mathcal{S}_{12} \\ \mathcal{S}_{21} - z^2(\mathcal{S}_{11} - \mathcal{S}_{22}) - z^4 \mathcal{S}_{12} & \mathcal{S}_{22} + z^2 \mathcal{S}_{12} \end{pmatrix}. \quad (2.56)$$

## 2.4 Relativistic top: $\eta$ -independent description

Consider the following Lax matrix

$$\tilde{L}(z, \tilde{S}) = \tilde{S}_0 1_{2 \times 2} + L(z, \tilde{S}) - \frac{1}{2} \text{tr} L(z, \tilde{S}) 1_{2 \times 2} \quad (2.57)$$

or

$$\tilde{L}(z) = \begin{pmatrix} \tilde{S}_0 & 0 \\ 0 & \tilde{S}_0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} \frac{1}{2}(\tilde{S}_{11} - \tilde{S}_{22}) - z^2 \tilde{S}_{12} & \tilde{S}_{12} \\ \tilde{S}_{21} - z^2(\tilde{S}_{11} - \tilde{S}_{22}) - z^4 \tilde{S}_{12} & \frac{1}{2}(\tilde{S}_{22} - \tilde{S}_{11}) + z^2 \tilde{S}_{12} \end{pmatrix}. \quad (2.58)$$

It consists of  $\mathfrak{sl}_2$  part of (2.28) and additional generator  $\tilde{S}_0$ . The Poisson structure is again the quadratic:

$$\{\tilde{L}_1(z), \tilde{L}_2(w)\} = [\tilde{L}_1(z) \tilde{L}_2(w), r_{12}(z-w)], \quad (2.59)$$

The classical Sklyanin algebra has the form:

$$\mathcal{A}_{h=0, \eta=0}^{\text{SkI}} : \quad \{\tilde{S}_1, \tilde{S}_2\} = \tilde{S}_0 [\tilde{S}_2, P_{12}] + [\tilde{S}_1 \tilde{S}_2, r_{12}^{(0)}] + [\text{tr}_3(r_{13}^{(0)} \tilde{S}_3) \tilde{S}_2, P_{12}]. \quad (2.60)$$

Substituting  $r^{(0)} = 0$  we get

$$\mathcal{A}_{h=0, \eta=0}^{\text{SkI}} : \quad \{\tilde{S}_1, \tilde{S}_2\} = \tilde{S}_0 [\tilde{S}_2, P_{12}]. \quad (2.61)$$

and

$$\{\tilde{S}_0, \tilde{S}\} = \lim_{\eta \rightarrow 0} \frac{[\tilde{S}, J^\eta(\tilde{S})]}{\eta} = [\tilde{S}, J(\tilde{S})], \quad (2.62)$$

i.e. the brackets between  $\mathfrak{sl}_2$ -variables keep the same form as in (2.21) but multiplied by  $\tilde{S}_0$ , while the brackets between any of  $\mathfrak{sl}_2$ -variables and  $\tilde{S}_0$  are just the corresponding non-relativistic equations of motion (2.24):

$$\begin{aligned} \{\tilde{S}_0, \tilde{S}_{11}\} &= -\{\tilde{S}_0, \tilde{S}_{22}\} = -\tilde{S}_{12}(\tilde{S}_{11} - \tilde{S}_{22}), \quad \{\tilde{S}_0, \tilde{S}_{12}\} = 2\tilde{S}_{12}^2, \\ \{\tilde{S}_0, \tilde{S}_{21}\} &= -2\tilde{S}_{12}\tilde{S}_{21} + (\tilde{S}_{11} - \tilde{S}_{22})^2, \end{aligned} \quad (2.63)$$

$$\{\tilde{S}_{11}, \tilde{S}_{12}\} = -\tilde{S}_0\tilde{S}_{12}, \quad \{\tilde{S}_{11}, \tilde{S}_{21}\} = \tilde{S}_0\tilde{S}_{21}, \quad \{\tilde{S}_{12}, \tilde{S}_{21}\} = \tilde{S}_0(\tilde{S}_{22} - \tilde{S}_{11}).$$

In other words, to get the quadratic algebra we add the Hamiltonian of the top to the  $\mathfrak{sl}_2$  Lie algebra generators. The Casimir functions are generated by

$$\det \tilde{L}(z) = \frac{1}{z^2} \tilde{C}_2 + \tilde{C}_0, \quad (2.64)$$

$$\tilde{C}_2 = -\frac{1}{4} (\tilde{S}_{11} - \tilde{S}_{22})^2 - \tilde{S}_{12}\tilde{S}_{21}, \quad \tilde{C}_0 = \tilde{S}_0^2 + 2\tilde{S}_{12}(\tilde{S}_{11} - \tilde{S}_{12}) \stackrel{(2.23)}{=} \tilde{S}_0^2 - 2H^{\text{top}}(\tilde{S}). \quad (2.65)$$

The Lax matrix is also the same (up to the scalar component). Therefore, the Lax pair is the same.

**Relation to  $\eta$ -dependent description.** The Lax matrices in  $\eta$ -dependent and  $\eta$ -independent descriptions  $L^\eta(z, S)$  and  $\tilde{L}(z, \tilde{S})$  are related as follows (2.7):

$$L^\eta(z + \eta_0, \tilde{L}(-\eta_0, S)) = \phi^\eta(z + \eta_0) \tilde{L}(z, S), \quad (2.66)$$

where

$$\phi^\eta(z) = \frac{\text{tr} L^\eta(z, S)}{\text{tr} S} \quad (2.67)$$

and

$$\eta_0 = \eta_0(\eta) : \quad \text{tr} L^\eta(\eta_0, S) = 0. \quad (2.68)$$

In  $\mathfrak{gl}_2$  case (2.46) we have

$$\phi^\eta(z) = \frac{2z + \eta}{z\eta}, \quad \eta_0 = -\eta/2. \quad (2.69)$$

The change of variables can be fixed as in (2.8)

$$\mathcal{S}(\eta, \tilde{S}) = \frac{1}{2} \tilde{L}\left(\frac{\eta}{2}, \tilde{S}\right). \quad (2.70)$$

These formulae allows us to pass from  $\mathcal{S}$  to  $\tilde{S}$  (2.58)

$$\begin{aligned} \tilde{S}_0 &= \text{tr} \mathcal{S} = \mathcal{S}_{11} + \mathcal{S}_{22}, \quad \tilde{S}_{11} - \tilde{S}_{22} = \eta(\mathcal{S}_{11} - \mathcal{S}_{22}) + \frac{1}{2}\eta^3 \mathcal{S}_{12}, \\ \tilde{S}_{12} &= \eta \mathcal{S}_{12}, \quad \tilde{S}_{21} = \eta \mathcal{S}_{21} + \frac{1}{4}\eta^3(\mathcal{S}_{11} - \mathcal{S}_{22}) + \frac{3}{16}\eta^5 \mathcal{S}_{12}. \end{aligned} \quad (2.71)$$

Notice that transition from the  $\eta$ -dependent Poisson structure (2.48)-(2.51) to the  $\eta$ -independent (2.61)-(2.63) can be also performed by taking the limit  $\eta \rightarrow 0$ . This is due to the structure of the change of variables (2.70), which contains a simple pole in  $\eta$  (at  $\eta = 0$ ).

## 2.5 Ruijsenaars-Schneider and Calogero-Moser models

The gauge transformations relating the rational Lax operators of the RS (CM) and the relativistic (non-relativistic) top models can be found in [20] ([1]). For the RS model

$$L^{\text{RS}}(z) = g^{-1}(z) g(z + \eta) e^{P/c} \rightarrow L^\eta(z) = g(z) L^{\text{RS}}(z) g^{-1}(z) = g(z + \eta) e^{P/c} g^{-1}(z). \quad (2.72)$$

In view of (1.2) the RS Lax matrix acquires the form

$$L^{\text{RS}}(z) = \text{tr}_2 \left( g_1^{-1}(z) \left( \text{Res}_{z=0} g_1^{-1}(z) \right) R_{12}^\eta(z) g_1(z) g_2(\eta) e^{P_2/c} \right). \quad (2.73)$$

Here we discuss the resultant change of variables (or bosonization formulae) given in  $\mathfrak{gl}_N$  case by

$$\mathcal{S}_{ij}(\mathbf{q}, \mathbf{p}) = \sum_{m=1}^N \frac{(q_m + \eta)^{\varrho(i)} e^{p_m/c}}{\prod_{k \neq m} (q_m - q_k)} (-1)^{\varrho(j)} \sigma_{\varrho(j)}(\mathbf{q}), \quad (2.74)$$

where  $\varrho(i) = i-1$  for  $i \leq N-1$  and  $\varrho(N) = N$ , while  $\sigma_k$  are the elementary symmetric functions.

In the center of mass frame set  $p_2 = -p_1 = -p$  and  $q_2 = -q_1 = -q$ , i.e. we deal with one pair of canonical coordinates

$$\{p, q\} = 1. \quad (2.75)$$

**RS model from  $\eta$ -dependent description.** The change of variables (2.74) with

$$\sigma_0(q) = -\frac{1}{4}(q_1 - q_2)^2 = -q^2, \quad \sigma_2(q) = 1$$

gives

$$\begin{aligned} \mathcal{S}_{11}(p, q) &= -\frac{q}{2} (e^{p/c} - e^{-p/c}), \quad \mathcal{S}_{12}(p, q) = \frac{1}{2q} (e^{p/c} - e^{-p/c}), \\ \mathcal{S}_{21}(p, q) &= -\frac{q}{2} (e^{p/c}(q - \eta)^2 - e^{-p/c}(q + \eta)^2), \\ \mathcal{S}_{22}(p, q) &= \frac{1}{2q} (e^{p/c}(q - \eta)^2 - e^{-p/c}(q + \eta)^2). \end{aligned} \quad (2.76)$$

In particular, it means that the Poisson brackets (2.51) follows from (2.76) and (2.75). Notice that having the factor  $1/c$  in the exponents one should also put it (as a common factor) to the r.h.s. of the Sklyanin algebra (2.50). To get it from the initial  $r$ -matrix structure (2.47) one can multiply the  $r$ -matrix by  $1/c$ .

The relativistic top's Hamiltonian equals

$$\text{tr} \mathcal{S}(p, q) = \mathcal{S}_{11}(p, q) + \mathcal{S}_{22}(p, q) = \eta \frac{\eta - 2q}{2q} e^{p/c} - \eta \frac{\eta + 2q}{2q} e^{-p/c}. \quad (2.77)$$

The RS Hamiltonian is proportional to  $\text{tr} \mathcal{S}(p, q)$ . Let us define it as

$$H^{\text{RS}} = -\eta^{-1} \text{tr} \mathcal{S}(p, q) = \frac{2q - \eta}{2q} e^{p/c} + \frac{2q + \eta}{2q} e^{-p/c}. \quad (2.78)$$

The passage to  $\tilde{S}$  variables can be made via (2.71).

**RS model in  $\eta$ -independent description** comes from the  $\eta$ -dependent (2.76) one and the change of variables (2.71). Plugging (2.76) into (2.71) we get

$$\begin{aligned}\tilde{S}_0 &= \eta \frac{\eta - 2q}{2q} e^{p/c} - \eta \frac{\eta + 2q}{2q} e^{-p/c}, \\ \tilde{S}_{11} - \tilde{S}_{22} &= -\frac{\eta}{4q} (e^{p/c}(2q - \eta)^2 - e^{-p/c}(2q + \eta)^2), \quad \tilde{S}_{12}(p, q) = \frac{\eta}{2q} (e^{p/c} - e^{-p/c}), \\ \tilde{S}_{21}(p, q) &= -\frac{\eta}{32q} (e^{p/c}(2q - \eta)^4 - e^{-p/c}(2q + \eta)^4).\end{aligned}\tag{2.79}$$

The RS Hamiltonian (2.78) obviously has the form

$$H^{\text{RS}} = -\eta^{-1} \tilde{S}_0.\tag{2.80}$$

**Calogero-Moser model** appears in the non-relativistic limit

$$\eta := \nu/c, \quad c \rightarrow \infty.\tag{2.81}$$

For the Hamiltonian (2.78) we have

$$H^{\text{RS}} = 2 + \frac{2}{c^2} H^{\text{CM}} + o\left(\frac{1}{c^2}\right),\tag{2.82}$$

where

$$H^{\text{CM}} = \frac{1}{2} p^2 - \nu \frac{p}{2q} = \frac{1}{2} \left( p - \frac{\nu}{2q} \right)^2 - \frac{1}{2} \frac{\nu^2}{(2q)^2}.\tag{2.83}$$

The conventional form  $H^{\text{CM}} = \frac{p^2}{2} + \frac{\nu^2}{(2q)^2}$  can be obtained by the substitution  $\nu \rightarrow \sqrt{-2}\nu$  and the canonical map  $p \rightarrow p + \frac{\nu}{2q}$ .

Consider also the limit (2.81) of the residue matrix  $\mathcal{S}$  (2.76):

$$S = -\frac{1}{2} \lim_{c \rightarrow \infty} c \mathcal{S} = \begin{pmatrix} \frac{1}{2} p q & -\frac{1}{2} \frac{p}{q} \\ \frac{1}{2} (p q^3 - 2\nu q^2) & -\frac{1}{2} p q + \nu \end{pmatrix}\tag{2.84}$$

The Poisson brackets between the matrix elements of (2.84) are the linear Poisson-Lie (2.21). The eigenvalues of the matrix  $S$  are equal to 0 and  $\nu$ .

Notice that the equations of motion of the relativistic top in the  $\eta$ -independent description have the same form as the non-relativistic equations (2.13). Therefore, equations (2.24)-(2.25) describe both - the 2-body RS model via the change of variables (2.79) and the 2-body CM model via (2.84).

The obtained formulae for  $\mathcal{S}(p, q)$  and  $S(p, q)$  are particular cases of those obtained in [20] and [1] at the level of classical mechanics. The underlying construction is the Symplectic Hecke Correspondence [16] (see also reviews [19, 27]). In fact, in the quantum counterpart of the quantum-classical relation (1.2) (i.e. from the Sklyanin algebra point of view) these type of formulae are known from the original papers [23] (in the elliptic case).

### 3 Spin chains and Gaudin models

#### 3.1 Gaudin models

Consider the phase space consisting of  $n$  copies of (2.20)-(2.22), i.e. the direct product of coadjoint orbits of  $GL_2$ . It means that we deal with  $S^a$ ,  $a = 1 \dots n$  and direct sum of the Poisson-Lie brackets

$$\{S_1^a, S_2^b\} = \delta^{ab} [S_2^a, P_{12}]. \quad (3.1)$$

Fixation of the Casimir functions  $C_{1,2}^a$  leaves 2-dimensional space for each  $S^a$ . Hence, the total dimension of the phase space is equal  $2n$ . Consider the Lax matrix written in terms of (2.28):

$$L^G(z) = \sum_{a=1}^n L(z - z_a, S^a). \quad (3.2)$$

The Hamiltonians appear by evaluating

$$\frac{1}{2} \text{tr} (L^G(z))^2 = \frac{1}{2} \sum_{a=1}^n \frac{\text{tr} (S^a)^2}{(z - z_a)^2} - \frac{h_a}{z - z_a} + 2h_0 \quad (3.3)$$

The direct computation gives

$$h_a = \sum_{c \neq a}^n h_{a,c}, \quad h_{a,c} = -\text{tr} (S^a L(z_a - z_c, S^c)) = -\text{tr}_{12} (r_{12}(z_a - z_c) S_1^a S_2^c), \quad (3.4)$$

or explicitly

$$h_{a,c} = -\frac{\text{tr}(S^a S^c)}{z_a - z_c} + (z_a - z_c) (S_{12}^a (S_{11}^c - S_{22}^c) + S_{12}^c (S_{11}^a - S_{22}^a)) + (z_a - z_c)^3 S_{12}^a S_{12}^c. \quad (3.5)$$

and

$$\begin{aligned} h_0 &= \frac{1}{2} \sum_{b,c=1}^n \text{tr} (S^b \mathcal{M}(z_b - z_c, S^c)) = - \sum_{b,c=1}^n S_{12}^b (S_{11}^c - S_{22}^c) + S_{12}^b S_{12}^c (z_b - z_c)^2 = \\ &= - \sum_{a=1}^n S_{12}^a (S_{11}^a - S_{22}^a) - \sum_{b>c}^n S_{12}^b (S_{11}^c - S_{22}^c) + S_{12}^c (S_{11}^b - S_{22}^b) + 2S_{12}^b S_{12}^c (z_b - z_c)^2, \end{aligned} \quad (3.6)$$

where  $\mathcal{M}(z, S)$  is the  $M$ -operator (2.33) with properties (2.35), (2.36).

The Hamiltonians (3.4)-(3.6) generate equations of motion

$$\begin{cases} \partial_{t_a} S^b = -[S^b, L(z_a - z_b, S^a)], & b \neq a = 1, \dots, n, \\ \partial_{t_a} S^a = \sum_{c \neq a}^n [S^a, L(z_c - z_a, S^c)], & a = 1, \dots, n \end{cases} \quad (3.7)$$

and

$$\partial_{t_0} S^a = [S^a, J(S^a)] + \sum_{c \neq a} [S^a, \mathcal{M}(z_a - z_c, S^c)], \quad (3.8)$$



where we used (2.35). Equations (3.7) and (3.8) have the Lax form

$$\partial_{t_d} L^G(z) = [L^G(z), M_d^G], \quad d = 0, 1, \dots, n \quad (3.9)$$

with

$$M_a^G(z) = -L(z - z_a, S^a), \quad a = 1, \dots, n \quad (3.10)$$

and

$$M_0^G(z) = \sum_{c=0}^n \mathcal{M}(z - z_c, S^c), \quad (3.11)$$

where  $\mathcal{M}(z, S)$  is from (2.33). Expression for  $M_d^G(z)$  can be obtained from the classical  $r$ -matrix structure which is the same as in the top case (2.38):

$$\{L_1^G(z), L_2^G(w)\} = [L_1^G(z) + L_2^G(w), r_{12}(z - w)] \quad (3.12)$$

with the  $r$ -matrix (2.39). The latter holds because the  $r$ -matrix structure is linear as well as the Poisson brackets (3.1).

The flows generated by  $h_a$  are not independent since

$$\sum_{c=1}^n h_c = 0. \quad (3.13)$$

Put it differently,

$$-\sum_{c=1}^n M_c^G(z) = L^G(z). \quad (3.14)$$

The total number of independent integrals of motion equals  $n$  ( $n - 1$  independent  $h_c$  and  $h_0$ ). This coincides with the half of dimension of the phase space. Hence the model is Liouville integrable.

In the limit (1.11)-(1.13)

$$L^G(z) \xrightarrow{\epsilon \rightarrow 0} \sum_{a=1}^n \frac{S^a}{z - z_a}, \quad (3.15)$$

the inverse inertia tensor  $J(S^a) \rightarrow 0$  (and  $\mathcal{M}(z, S) \rightarrow 0$ ) and the Hamiltonian  $h_0$  (3.6) become trivial. In the same time the isotropic limit restores the common  $GL_2$  symmetry. It compensates the lost of one Hamiltonian.

Let us mention that in the light of the Symplectic Hecke Correspondence [16] the obtained Gaudin model are gauge equivalent to those considered in [21]. An explicit relation to [21] requires some gauge fixation related to additional reduction by the (global) Cartan subgroup coadjoint action. The latter action is a common feature of the models with the dynamical  $r$ -matrices.

### 3.2 Spin chains and reflection equations

The classical periodic spin chain on  $n$  sites is constructed by introducing the transfer matrix [8]:

$$T(z) = L^n(S^1, z - z_1) \dots L^n(S^n, z - z_n). \quad (3.16)$$

The Lax operators which satisfy the quadratic Poisson relation (2.2) or (2.5) The transfer matrix also satisfies the quadratic Poisson relations. Depending on the choice of description we have the quasi-classical or purely classical expression:

$$T_0(z) = \text{tr}_{1\dots n} (R_{01}^{\eta_1}(z - z_1) \dots R_{0n}^{\eta_n}(z - z_n) (\mathcal{S}^1)_1 \dots (\mathcal{S}^n)_n) \quad (3.17)$$

or

$$\tilde{T}_0(z) = \text{tr}_{1\dots n} (r_{01}(z - z_1) \dots r_{0n}(z - z_n) \tilde{S}_1^1 \dots \tilde{S}_n^n) . \quad (3.18)$$

The classical local Hamiltonian appears as follows: set  $z_k = 0$  and let the Casimir functions (2.55) or (2.65) be equal for all the sites. Then one should compute the transfer-matrix at point  $z_0$  given by condition  $\det L^k(z_0) = 0$  (it holds simultaneously for all sites due to above requirements).

To get the spin chain on the finite lattice we also need another Poisson algebra (classical reflection equation) at the boundaries [24]:

$$\begin{aligned} \{\tilde{L}_1(z), \tilde{L}_2(w)\} = \\ \frac{1}{2}[\tilde{L}_1(z) \tilde{L}_2(w), r_{12}(z - w)] - \frac{1}{2} \tilde{L}_1(z) r_{12}(z + w) \tilde{L}_2(w) + \frac{1}{2} \tilde{L}_2(w) r_{12}(z + w) \tilde{L}_1(z), \end{aligned} \quad (3.19)$$

It appears by reduction from (2.5) using the constraints

$$\tilde{L}(z, \tilde{S}) \tilde{L}(-z, \tilde{S}) = \det \tilde{L}(z, \tilde{S}) 1_{2 \times 2} . \quad (3.20)$$

The Lax matrix (2.58) satisfies this condition.

One can verify the following statement:

*The classical  $\eta$ -independent Lax operator (2.58) satisfies the reflection equation (3.19). The resultant Poisson brackets for the components of  $\tilde{S}$  coincide with (2.61)-(2.62).*

It means that we add for the boundaries two Lax operators  $L^\pm(z, S^\pm)$ . They are described by the Lax matrices and satisfy the same classical algebra (2.61)-(2.62) but through the reflection equation (3.19). The spin chain transfer-matrix (with dynamical boundaries) is then defined as

$$T(z) = \tilde{L}(S^+, z) \tilde{L}(S^1, z - z_1) \dots \tilde{L}(S^n, z - z_n) \tilde{L}(S^-, z) \tilde{L}(S^n, z - z_n) \dots \tilde{L}(S^1, z - z_1) . \quad (3.21)$$

Remark that at the boundaries we use the same Lax operators as inside the chain. In the elliptic case there is an opportunity to put the extended Lax operators, which give rise to the inhomogeneous Sklyanin algebra. It is related to the  $BC_1$  elliptic model. The corresponding mechanical model has form of the gyrostat [17].

At last, notice that all the consideration can be performed in terms of the  $\eta$ -dependent Lax matrix (2.46). In particular, due to (2.7) we have the following form of the reflection equation for  $L^\eta(z)$ :

$$\begin{aligned} \{L_1^\eta(z), L_2^\eta(w)\} = \frac{1}{2}[L_1^\eta(z) L_2^\eta(w), r_{12}(z - w)] - \\ - \frac{1}{2} L_1^\eta(z) r_{12}(z + w + \eta) L_2^\eta(w) + \frac{1}{2} L_2^\eta(w) r_{12}(z + w + \eta) L_1^\eta(z) . \end{aligned} \quad (3.22)$$

### 3.3 Canonical variables and many-body interpretation

The described Gaudin models and spin chains (as well as their XXX limits) can be rewritten in the canonical coordinates of the 2-body CM model (2.84) or the 2-body RS model (2.76), (2.79). It differs from the standard parametrization of  $\mathfrak{sl}_2^*$  which leads to the Garnier type models [13].

Consider first the Gaudin model. Let all residues  $S^a$  are parameterized by the canonical coordinates  $S^a = S^a(p_a, q_a, \nu_a)$

$$\{p_a, q_b\} = \delta_{ab}, \quad a, b = 1, \dots, n, \quad (3.23)$$

where  $n$  is the number of the sites (poles in the Gaudin Lax matrix). Plugging  $S^a(p_a, q_a, \nu_a)$  given by (2.84) into the Gaudin Lax matrix (3.2) we get

$$L^G(z) = \sum_{a=1}^n L(z - z_a, S^a(p_a, q_a, \nu_a)). \quad (3.24)$$

an integrable  $n$ -particle integrable system depending on  $2n$  constants  $\nu_a$  and  $z_a$ . The same, of course, can be done for the XXX limit (3.15). For example,

$$\text{tr}(S^a S^b) = \left( \frac{p_a}{2q_a} (q_b^2 - q_a^2) + \nu_a \right) \left( \frac{p_b}{2q_b} (q_a^2 - q_b^2) + \nu_b \right). \quad (3.25)$$

Rewriting in this way the Gaudin Hamiltonians (3.4)-(3.6) we get

$$\begin{aligned} h_a = & - \sum_{c \neq a} \frac{1}{z_a - z_c} \left( \frac{p_a}{2q_a} (q_c^2 - q_a^2) + \nu_a \right) \left( \frac{p_c}{2q_c} (q_a^2 - q_c^2) + \nu_c \right) + \\ & + \frac{z_a - z_c}{2} \left( \frac{p_a}{q_a} (p_c q_c - \nu_c) + \frac{p_c}{q_c} (p_a q_a - \nu_a) \right) - (z_a - z_c)^3 \frac{p_a p_c}{4q_a q_c} \end{aligned} \quad (3.26)$$

and

$$h_0 = \sum_{a=1}^n \frac{p_a}{2q_a} (p_a q_a - \nu_a) + \frac{1}{2} \sum_{b>c} \left( \frac{p_b}{q_b} (p_c q_c - \nu_c) + \frac{p_c}{q_c} (p_b q_b - \nu_b) - (z_b - z_c)^2 \frac{p_b p_c}{2q_b q_c} \right). \quad (3.27)$$

The first sum in (3.27) equals the sum of 2-body CM Hamiltonians  $\sum H^{CM}(p_a, q_a, \nu_a)$  (2.83). Therefore, this Hamiltonian describes  $n$  particles with masses  $m_j \sim \nu_j^2$  in the central field  $\sim m_j/q_j^2$  with additional non-trivial interaction. Notice that this system depends on  $2n$  free parameters  $\{z_a\}$ ,  $\{\nu_a\}$ . The deformation parameter  $\epsilon$  (1.13) can be added as well. In the XXX limit the Hamiltonian  $h_0$  vanishes, and only the upper line of (3.26) survives for  $h_a$ .

Similar calculations can be made for the spin chain (3.17) or (3.18). One can use parametrization in canonical (RS) variables  $\mathcal{S}^a(p_a, q_a, \eta_a)$  (2.76) or  $\tilde{S}^a(p_a, q_a, \eta_a)$  (2.79) respectively. For example, similarly to (3.25)

$$\begin{aligned} \text{tr}(\mathcal{S}^a(p_a, q_a, \eta_a) \mathcal{S}^b(p_b, q_b, \eta_b)) = & \left[ \frac{q_a}{2} (e^{p_b/c} - e^{-p_b/c}) - \frac{1}{2q_a} (e^{p_b/c} (q_b - \eta_b)^2 - e^{-p_b/c} (q_b + \eta_b)^2) \right] \times \\ & \left[ \frac{q_b}{2} (e^{p_a/c} - e^{-p_a/c}) - \frac{1}{2q_b} (e^{p_a/c} (q_a - \eta_a)^2 - e^{-p_a/c} (q_a + \eta_a)^2) \right]. \end{aligned} \quad (3.28)$$

When  $\eta_a = \eta_b = \eta$  this can be used for rewriting the XXX local Hamiltonian  $\sum_k \text{tr}(\mathcal{S}^k \mathcal{S}^{k+1})$ .

## 4 1+1 models

Here we consider the models, which are integrable in the sense of existing of the Zakharov-Shabat equations [31]:

$$\partial_t U(z) - k \partial_x V(z) = [U(z), V(z)], \quad (4.1)$$

where  $x$  is a coordinate on the circle. The dynamical variables become the periodic fields with the Poisson brackets:

$$\{S_{ij}(x), S_{kl}(y)\} = (S_{kj}(x)\delta_{il} - S_{il}(x)\delta_{kj}) \delta(x - y). \quad (4.2)$$

We keep notation  $S(x) = S$ . The procedure of 1+1 generalization of the models described by non-dynamical  $r$ -matrices is simple (in contrast to the case of dynamical  $r$ -matrices related to many-body systems, see [16]) – one should use the same Lax matrix ( $U$  matrix) as in the mechanical (top) case. The problem of finding  $V$  in general case is more complicated. In the cases under consideration we will use the ansatz from [25] and its natural generalization [32].

### 4.1 Landau-Lifshitz equation

Set  $S_{22}(x) = -S_{11}(x)$  and let  $S^2 = \lambda^2 1$ ,  $\partial_x \lambda = 0$ . Consider the U-V pair:

$$U^{\text{LL}} = L(z, S(x)) = \frac{1}{z} \begin{pmatrix} S_{11} - z^2 S_{12} & S_{12} \\ S_{21} - 2z^2 S_{11} - z^4 S_{12} & -S_{11} + z^2 S_{12} \end{pmatrix} \quad (4.3)$$

and

$$V^{\text{LL}} = -\frac{1}{2}(V_1^{\text{LL}} + V_2^{\text{LL}}) \quad (4.4)$$

$$V_1^{\text{LL}} = \frac{1}{z} L(z, S) - 2\mathcal{M}(z, S) = \frac{1}{z^2} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & -S_{11} \end{pmatrix} + \begin{pmatrix} S_{12} & 0 \\ 2S_{11} + 3z^2 S_{12} & -S_{12} \end{pmatrix}, \quad (4.5)$$

where  $\mathcal{M}$  is from (2.33),

$$V_2^{\text{LL}} = L(z, h) = \frac{1}{z} \begin{pmatrix} h_{11} - z^2 h_{12} & h_{12} \\ h_{21} - 2z^2 h_{11} - z^4 h_{12} & -h_{11} + z^2 h_{12} \end{pmatrix} \quad (4.6)$$

where the matrix  $h$  equals

$$h = -\frac{k}{4\lambda^2} [S, S_x], \quad S_x = \partial_x S. \quad (4.7)$$

Plugging this U-V pair into (4.1) we get two equations:

$$\begin{aligned} -k \partial_x V_1^{\text{LL}} &= [L, V_2^{\text{LL}}], \\ \partial_t L + \frac{1}{2} k \partial_x V_2^{\text{LL}} &= -\frac{1}{2} [L, V_1^{\text{LL}}] = [L, \mathcal{M}] \end{aligned} \quad (4.8)$$

and, hence

$$-k\partial_x S = [S, h], \quad (4.9)$$

$$\partial_t S + (k/2)\partial_x h = [S, J(S)].$$

Due to the relation  $SS_x + S_x S = 0$  the first equation can be solved as given in (4.7). Then the second equation assumes the form:

$$\partial_t S = \alpha[S, S_{xx}] + [S, J(S)] \quad (4.10)$$

with the constant  $\alpha = k^2/8\lambda^2$ . In components we have (cf.(2.24)):

$$\begin{cases} \partial_t S_{11} = \alpha S_{12} \partial_x^2 S_{21} - \alpha S_{21} \partial_x^2 S_{12} - 2S_{12} S_{11}, \\ \partial_t S_{21} = 2\alpha S_{21} \partial_x^2 S_{11} - 2\alpha S_{11} \partial_x^2 S_{21} - 2S_{12} S_{21} + 4S_{11}^2, \\ \partial_t S_{12} = 2\alpha S_{11} \partial_x^2 S_{12} - 2\alpha S_{12} \partial_x^2 S_{11} + 2S_{12}^2. \end{cases} \quad (4.11)$$

The Hamiltonian equals

$$H^{\text{LL}} = \frac{1}{2} \oint dx \left( \text{tr}(S_x^2) + \text{tr}(SJ(S)) \right). \quad (4.12)$$

The limit (1.12) to the continuous Heisenberg model [29] can be performed as in the finite-dimensional case (see (3.15) for  $n = 1$ ). Therefore, we obtained an integrable deformation of the Heisenberg model. Let us mention that close (but different) rational Landau-Lifshitz equations were found recently in the context of AdS/CFT correspondence [9].

## 4.2 Principal chiral model

To get the (anisotropic) principal chiral model [30, 6, 8] consider the phase space

$$\{S_{ij}^a(x), S_{kl}^b(y)\} = \delta^{ab} (S_{kj}^a(x)\delta_{il} - S_{il}^a(x)\delta_{kj}) \delta(x - y), \quad a, b = 1, 2. \quad (4.13)$$

and set

$$L^1 = L(z - z_1, S^1(x)), \quad L^2 = L(z - z_2, S^2(x)) \quad (4.14)$$

with  $L(z, S)$  (2.28). Then the U-V pair

$$\begin{cases} U^{\text{chiral}} = L^1 + L^2, \\ V^{\text{chiral}} = L^1 - L^2. \end{cases} \quad (4.15)$$

for the Zakharov-Shabat equation (4.1) gives

$$\begin{cases} \partial_t S^1 - k\partial_x S^1 = -2[S^1, L(z_1 - z_2, S^2)], \\ \partial_t S^2 + k\partial_x S^2 = -2[L(z_2 - z_1, S^1), S^2]. \end{cases} \quad (4.16)$$

This is the rational analogue of the anisotropic model [6]. To see its relation to the isotropic one, consider the deformation (1.12)

$$L_\epsilon(z_1 - z_2, S) = \frac{1}{z_1 - z_2} S + \delta_\epsilon L, \quad (4.17)$$

where

$$\delta_\epsilon L = - \begin{pmatrix} \epsilon^2(z_1 - z_2)S_{12} & 0 \\ 2\epsilon^2(z_1 - z_2)S_{11} + \epsilon^4(z_1 - z_2)^3S_{12} & -\epsilon^2(z_1 - z_2)S_{12} \end{pmatrix} \quad (4.18)$$

In the isotropic (XXX) limit  $\epsilon \rightarrow 0$  we find  $L^1 = S^1/(z - z_1)$ ,  $L^2 = S^2/(z - z_2)$ . Then, by setting  $S^\pm = S^1 \pm S^2$  one gets the conventional form of the principal chiral model:

$$\begin{cases} \partial_t S^- - k \partial_x S^+ = [S^-, S^+], \\ \partial_t S^+ - k \partial_x S^- = 0. \end{cases} \quad (4.19)$$

It is remarkable that in [6] the author obtained the anisotropic chiral model starting from the 1-site XYZ model, i.e. from the one pole case instead of the two-poles ansatz (4.14). This can be explained by the passage to the light-cone coordinates

$$\xi = \frac{kt + x}{2k}, \quad \eta = \frac{kt - x}{2k}. \quad (4.20)$$

Taking into account the skew-symmetry  $L(z) = -L(-z)$ , (4.16) acquires the form:

$$\begin{cases} \partial_\eta S^1 = -2[S^1, L(z_1 - z_2, S^2)], \\ \partial_\xi S^2 = -2[S^2, L(z_1 - z_2, S^1)], \end{cases} \quad (4.21)$$

Then, the reduction  $\partial_\xi S^2 = 0$  leads to the equation  $[S^2, L(z_1 - z_2, S^1)] = 0$ . It has a particular solution  $S^2|_{red} = -\frac{1}{2}L(z_1 - z_2, S^1)$ . From (2.37)  $L(z_1 - z_2, S^2|_{red}) = -\frac{1}{2} \left( \frac{1}{(z_1 - z_2)^2} S^1 + 2J(S^1) \right)$ . Then, from the first equation in (4.21) we get the top equations  $\partial_\eta S^1 = [S^1, J(S^1)]$  (1.8). In this sense the equations (4.21) (and hence (4.16)) can be also considered as 1+1 generalization of the top described by the single pole  $z_1 = 0$  Lax matrix (1.7).

### 4.3 Interacting Landau-Lifshitz magnets

An arbitrary number ( $n$ ) of poles in  $U$

$$U = \sum_{a=1}^n L(z - z_a, S^a(x)) \quad (4.22)$$

gives rise to the 1+1 Gaudin type model. It was studied in [32]. The elliptic formulae obtained in that paper work for the rational case under consideration as well. It can be treated as the model of interacting Landau-Lifshitz magnetics in the same sense as the  $t_0$  flow of the Gaudin model (3.8) looks like interacting tops<sup>7</sup> (1.8). From the spin chain point of view these type of models arise from combining each  $n$  neighbor sites into one. The quantum and classical (Poisson) Sklyanin-type algebras underlying discrete version of 1+1 Gaudin model were described in [7].

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<sup>7</sup>There is another meaning for the "interacting tops" [18] coming from intermediate cases between the purely dynamical and purely non-dynamical  $R$ -matrices.

Equations of motion for the 1+1 Gaudin model are of the form:

$$\begin{cases} \partial_{t_a} S^a = \partial_x h^a + [S^a, J(S^a)] + \sum_{c \neq a} [h^a, L(z_c - z_a, S^c)] - [V_1^{\text{LL}}(z_c - z_a, S^c), S^a], \\ \partial_{t_a} S^b = [S^b, V_1^{\text{LL}}(z_b - z_a, S^a) - L(z_a - z_b, h^a)], \quad b \neq a, \end{cases} \quad (4.23)$$

where  $a, b = 1, \dots, n$ ,  $V_1^{\text{LL}}$  is given in (4.5) and

$$h^a = \alpha[S^a, \partial_x S^a] + \sum_{c \neq a} L(z_a - z_c, S^c). \quad (4.24)$$

When  $n = 1$  the model coincides with the Landau-Lifshitz one (4.10).

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